

Exact dark state solutions of the coupled atomic-molecular Bose-Einstein condensates in an external potential

Xiao-Fei Zhang^{1,3}, Jun-Chao Chen², Biao Li², Lin Wen¹, and W. M. Liu¹

¹*Beijing National Laboratory for Condensed Matter Physics,*

Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China

²*Nonlinear Science Center, Ningbo University, Ningbo 315211, China and*

³*College of Science, Honghe University, Mengzi 661100, China*

(Dated: August 26, 2011)

We consider a coupled nonlinear Schrödinger equations describing an atomic Bose-Einstein condensates coupled to a molecular condensates through the stimulated Raman adiabatic passage loaded in an external potential. The existence of dark state are investigated within the full parameter space accounts for all the nonlinear collisions, together with the atom-molecule conversion coupling and external potential. The results show that there are a class of external potentials such as double-well, periodical and double periodical, and harmonic potentials, where the exact dark solutions can be formed. Our results may raise the possibility of relative experiments for dark state in the coupled atomic-molecular Bose-Einstein condensates.

PACS numbers: 03.75.Mn, 03.75.Lm, 03.75.Hh, 05.45.Yv

I. INTRODUCTION

The experimental realization of Bose-Einstein condensates (BECs) [1–3] arouse great interest in ultracold molecule due to their potential applications for the tests of fundamental physics and for the drifts of fundamental constants [4–8]. This fundamental system is of special interest since the collisional coherence can led to remarkable demonstration of the reversible atom-molecule formation [9]; thus open the floodgate for many beautiful experiments, such as coherent oscillations between atoms and molecules, dissociation of molecular BECs, the “clumping” of the condensates due to the modulational instability, and so on [10–12]. It also represents a rich ground for investigations of the nonlinear excitation of the coupled atomic-molecular Bose-Einstein condensates (AMBEC) [13].

In real experiments, the cold molecules can be produced from a Fermi gas of atoms [14–16] or an atomic BEC based on Feshbach resonances, Raman photoassociation, or stimulated Raman adiabatic passage (STIRAP) [17–25]. However, among them, the Feshbach coupling mechanism is restricted to the creation of molecules in the highest rovibrational level and is only practicable for a limited number of systems [26–28], and the losses caused by inelastic atom-molecule collisions occurs at a significant rate. In the Raman photoassociation process, the effective conversion rate of a pair of atoms to ground molecule is limited by spontaneous emission from the excited molecular state; while the STIRAP is known to have the highest rate of efficiency when converting an atomic condensates into a molecular one based on a Raman transition, where the input Raman laser pulse couples the molecular levels and reduces spontaneous emission. During this process, the molecular state is in a dark superposition state (or coherent population trapping state, CPT state), which decouples from the light and thus suppresses losses due to spontaneous light

scattering [29]. In this situation, the excited molecular state of this coupled system remains almost unpopulated. Thus it is not difficult to see the existence of dark is the key ingredient in effective production of molecule through STIRAP technique.

Generally speaking, the dark state cannot exist with any external potential as the coupled nonlinear Schrödinger equations (see Eq. (1) below) is nonintegrable [30–35]. The main purpose of the present paper is to find the exact dark state solutions of this coupled system with tunable nonlinear interactions and external potential. Based on the framework of a three-component mean-field model that takes into account all types of the mean-field nonlinear atomic and molecular interactions and external potential. we present a comprehensive analysis of the dark states in the parametrically coupled atomic-molecular Bose-Einstein condensates. The results show that there are a class of external potentials, such as double-well, periodical and double periodical, and harmonic potentials, which can be used to support the existence of the dark states.

This paper is organized as follows. In sec. II, we briefly outline the theoretical model under study and the underlying mean-field coupled equations. Then a class of simplification is presented and the reduced coupled equations, focusing on the CPT solutions, are obtained. In Sec. III, the analytical methods for solving the coupled Schrödinger equations are introduced, and we present several classes of exact dark state solutions where different potentials can be applied to trap the coupled condensates and relationship between system parameters are established. A simple summary is given in the final section.

II. THE THEORY MODEL

To begin with, we consider a quasi-one-dimensional geometry of the condensates where the transverse traps are tight enough, thus the transverse motion of the condensates are frozen to the ground state of the transverse harmonic trapping potential. The coupled Heisenberg equations of motion for the atomic and molecular components are governed by the system of equations as follows:

$$\begin{aligned}
i\hbar \frac{\partial \psi_1}{\partial t} &= -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_1}{\partial x^2} + V_1 \psi_1 - \frac{\hbar \Omega_1}{\sqrt{2}} \psi_3 \psi_1^* \\
&\quad + (U_{11} |\psi_1|^2 + U_{12} |\psi_2|^2 + U_{13} |\psi_3|^2) \psi_1, \\
i\hbar \frac{\partial \psi_2}{\partial t} &= -\frac{\hbar^2}{4m} \frac{\partial^2 \psi_2}{\partial x^2} + V_2 \psi_2 - \frac{\hbar \Omega_2}{2} \psi_3 \\
&\quad + (U_{21} |\psi_1|^2 + U_{22} |\psi_2|^2 + U_{23} |\psi_3|^2) \psi_2, \\
i\hbar \frac{\partial \psi_3}{\partial t} &= -\frac{\hbar^2}{4m} \frac{\partial^2 \psi_3}{\partial x^2} + V_3 \psi_3 - \frac{\hbar \Omega_1}{2\sqrt{2}} \psi_1^2 - \frac{\hbar \Omega_2}{2} \psi_2 \\
&\quad + (U_{31} |\psi_1|^2 + U_{32} |\psi_2|^2 + U_{33} |\psi_3|^2) \psi_3, \quad (1)
\end{aligned}$$

here we define the order parameter ψ_a represents an atomic species of mass m in a potential $V_1(x)$; while ψ_2 and ψ_3 represents the ground and excited molecular species with a potential $V_2(x)$ and $V_3(x)$, respectively. The nonlinear atom-atom interaction $U_{11} = 4\pi\hbar^2 a_{11}/m$, the atom-molecule interaction $U_{12} = U_{21} = (3a_{12}/4a_{11})U_{11}$, and the ground molecule-molecule interaction $U_{22} = (a_{22}/2a_{11})U_{11}$, with a_{ij} being the corresponding s -wave scattering lengths modulated by a Feshbach resonance. Ω_1 is the Rabi frequency corresponding to the transitions between the atomic state and the excited molecular state, and Ω_2 is the one between the molecular ground and excited states. Finally, in this paper, we will only focus on the degenerate case with $V_2 = V_3 = 2V_1$ and ignore loss and growth mechanisms [35, 36]; thus the total number of atomic particles, including pairs of atoms inside the diatomic molecules is conserved in this model.

From a general mathematical and nonlinear physical point of view, Eq. (1) is an example of a classically non-integrable field theory, which needs to be treated numerically. To obtain the exact dark state solutions, we make a class of simplifications. To this end, we first neglect the excited molecular state as it has a small population compared with atomic and ground molecular states, and then set $U_{3i} = 0$. Similar to homogeneous condensates cases, the STIRAP model considered in this paper is also found to support a CPT state (with $\psi_3(x, t) = 0$) in the form of

$$\psi_1(x, t) = \phi_1(x)e^{-i\mu t}, \psi_2(x, t) = \phi_2(x)e^{-i2\mu t}. \quad (2)$$

Normalizing the time and length in Eq. (1) by $t' = \omega_2 t/2$ and $x' = \sqrt{\omega m/\hbar} x$ (the tilde is omitted for simplicity in the following discussions). Then inserting this

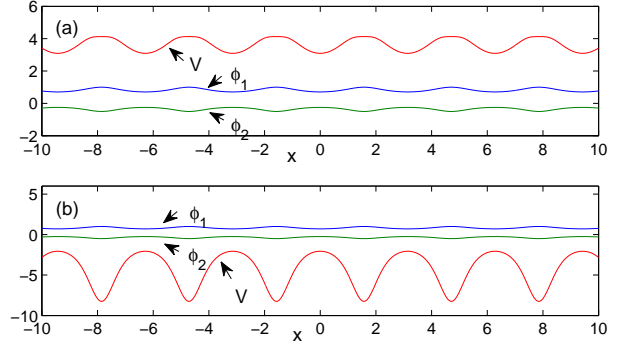


FIG. 1: (Color online) The stationary dark state wave functions of the atomic and molecular condensates $\phi_{11}(x)$ and $\phi_{21}(x)$ given by Eq. (10), and the profile of the external potential in the limit $k = 0$ and $a_0 = 0$. The parameters are given as follows: $\delta = 1, a_0 = 0, B_1 = a_1 = A_0 = \Omega = \lambda = \mu = 1$, and $g_{22} = 1$ in Fig. 1a and $g_{22} = 34$ in Fig. 1b. As shown in this figure, different external potentials but with similar periodical structure can support the same stationary dark state wave functions of the atomic and molecular condensates.

equation into Eq. (1), one can readily derive the following stationary equations:

$$\begin{aligned}
\mu \phi_1 &= -\frac{d^2 \phi_1}{dx^2} + (g_{11} \phi_1^2 + g_{12} \phi_2^2) \phi_1 + V(x) \phi_1, \\
\mu \phi_2 &= -\frac{1}{4} \frac{d^2 \phi_2}{dx^2} + \frac{1}{2} (g_{22} \phi_2^2 + g_{21} \phi_1^2) \phi_2 + V(x) \phi_2, \quad (3)
\end{aligned}$$

with $\phi_2 = -\Omega \phi_1^2/2$. Here we use the fact $\mu_2 = 2\mu_1 = 2\mu$ and the definition $\Omega = \Omega_1/\Omega_2$. In Ref. [35], Eq. (3) is reduced to a single ordinary differential equation and the possibilities of existence of inhomogeneous dark states of atomic-molecular Bose-Einstein condensates loaded in special external potential is discussed. In what follows we find that the exact dark state solutions can also exist in other types of external potentials, such as double-well, periodical and double periodical potentials.

III. EXACT DARK STATE SOLUTIONS AND DISCUSSION

In this section, we introduce an analytical method to derive a class of exact dark state solutions for Eq. (3). Without a trap, i.e., $V_i = 0$, the system is homogeneous and the dark state is well studied (see Refs. [30, 31, 37] and references therein). In the presence of external potential, we find that some special types of dark state solutions can be derived via suitable selections of the nonlinear parameters and external potential.

To get the analytical dark state solutions, we assume the stationary dark state solutions have the following forms:

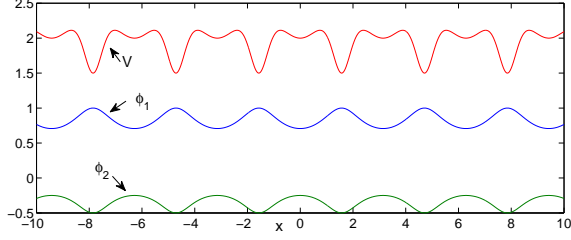


FIG. 2: (Color online) The stationary dark state wave functions of the atomic and molecular condensates $\phi_{11}(x)$ and $\phi_{21}(x)$ given by Eq. (10), and the profile of external potential in the limit $k = 0$ and $a_0 = B_0 = 0$. The other parameters are the same as in Fig (1) except $g_{22} = 8$. In this case, the shape of external potential shows double-periodical structure.

$$\begin{aligned}\phi_1(x) &= \delta \sqrt{a_0 + a_1 \varphi}, \\ \phi_2(x) &= -\frac{1}{2} \Omega (a_0 + a_1 \varphi),\end{aligned}\quad (4)$$

with auxiliary function $\varphi = \varphi(x)$ satisfying the following equations

$$\begin{aligned}\left(\frac{d\varphi}{dx}\right)^2 &= h_0 + h_1 \varphi + h_2 \varphi^2 + h_3 \varphi^3 + h_4 \varphi^4, \\ \frac{d^2 \varphi}{dx^2} &= \frac{h_1}{2} + h_2 \varphi + \frac{3}{2} h_3 \varphi^2 + 2 h_4 \varphi^3,\end{aligned}\quad (5)$$

where $\delta = \pm 1$ and $a_0, a_1, h_0, h_1, h_2, h_3, h_4$ are constants to be determined. Now we shift our attention to the external potential and further assume the external potential is in the form of

$$V = \frac{b_0 + b_1 \varphi + b_2 \varphi^2 + b_3 \varphi^3 + b_4 \varphi^4}{d_0 + d_1 \varphi}, \quad (6)$$

with $b_0, b_1, b_2, b_3, b_4, d_0, d_1$ are constants to be determined. Substituting Eqs. (4) and (6) into Eq. (3) and using the auxiliary Eq. (5), we can obtain a set of ordinary differential equations (ODEs) with respect to the controllable system parameters $\{\Omega, g_{11}, g_{12}, g_{22}\}$. As shown in Eqs. (4) and (5), depending on different choices of the form of $\varphi(x)$, we can get various types of external potential and exact dark state solutions through Eqs. (4) and (6). It should be mentioned that in order to simplify the calculation, we further set $\delta = 1$ in the following discussions. Finally, solving these ODEs, we can obtain two families of analytical solutions of Eq. (3).

A. The Jacobi cn-wave solutions

We begin with the Jacobi cn-function solution of $\varphi(x)$. In this case, we can get

$$\varphi_1 = \frac{A_0}{B_0 + B_1 \text{cn}^2(\lambda x, k)}, \quad (7)$$

where cn is the Jacobi elliptic function and $0 \leq k \leq 1$ is the module of Jacobi elliptic functions, $B_0 \neq -B_1$. At this point, the analytical dark state solutions for the atomic and molecular condensates can be written as

$$\begin{aligned}\phi_1 &= \sqrt{\frac{a_1 A_0}{B_0 + B_1 \text{cn}^2(\lambda x, k)}}, \\ \phi_2 &= \frac{-\Omega a_1 A_0}{2B_0 + 2B_1 \text{cn}^2(\lambda x, k)},\end{aligned}\quad (8)$$

here we set $a_0 = 0$. In this situation, the external potential can be obtained by solving Eq. (6) and reads

$$V_1(x) = \left(\frac{1}{2} \Omega^2 g_{12} - \frac{3}{8} \Omega^2 g_{22} \varphi_1^2(x) + (3g_{11} - 2g_{12}) \varphi_1(x) + \mu + \frac{h_2}{4}\right). \quad (9)$$

It is evident that the external potential is a function of nonlinear parameters, chemical potential, the Rabi frequencies, and the auxiliary function $\varphi(x)$. In the following discussions, we find that with different choices of the auxiliary function $\varphi(x)$, there exist a class of external potentials, such as double-well, periodical and double periodical, and harmonic potentials, where analytical dark state can exist and the constraint conditions for the nonlinear parameters can be obtained. For simplicity, we only discuss the exact dark state solutions of Eq. (3) in two cases with the module of Jacobi elliptic functions $k = 0$ and $k = 1$.

Case 1. In the limit $k = 0$, one can easily get

$$\begin{aligned}\phi_{11} &= \sqrt{\frac{a_1 A_0}{B_0 + B_1 \cos^2(\lambda x)}}, \\ \phi_{21} &= \frac{-\Omega a_1 A_0}{2B_0 + 2B_1 \cos^2(\lambda x)}.\end{aligned}\quad (10)$$

where $\lambda = \pm 1$. In this case, the relationship between interaction parameters read

$$\begin{aligned}g_{12} &= \frac{-4\lambda^2(B_1 B_0 + B_0^2)}{\Omega^2 a_1^2} + \frac{g_{22}}{2}, \\ g_{11} &= \frac{-2\lambda^2(B_1 B_0 + B_0^2)}{\Omega^2 a_1^2} + \frac{\lambda^2 B_0}{a_1} + \frac{B_1 \lambda^2}{2a_1} + \frac{g_{22}}{4}\end{aligned}\quad (11)$$

with $A_0 = 1$ and g_{22} a function of the system parameters.

Shown in Fig. 1 are the profiles of the wave functions for the atomic and molecular condensates $\phi_1(x)$ and $\phi_2(x)$, and the external potential in the limit $k = 0$. The parameters are given as $\delta = 1$, $B_1 = a_1 = A_0 = \Omega = 1$, $\lambda = 1$, and $g_{22} = 1$ in Fig. 1(a) and $g_{22} = 34$ in Fig. 1(b). According to Eq. (11), which gives the relationship among these three kinds of nonlinear interactions;

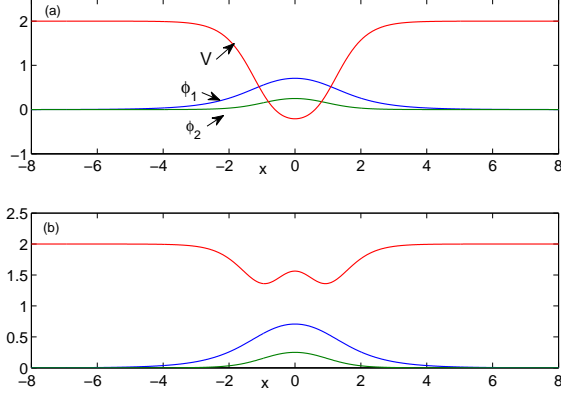


FIG. 3: (Color online) The stationary dark state wave functions of the atomic and molecular condensates $\phi_{12}(x)$ and $\phi_{22}(x)$ given by Eq. (12), and the profile of external potential in the limit $k = 1$. The parameters are given as follows: $\delta = 1$, $B_0 = B_1 = a_0 = 1$, $\lambda = 1$, $\Omega = -1$, and $g_{22} = -20/3$ in (a) and $g_{22} = -18$ in (b).

for Fig. 1(a), both the atom-atom and atom-molecule interactions $g_{11} = -2.25 < 0$ and $g_{12} = -7.5 < 0$, corresponding to attractive atom-atom and atom-molecule interactions; while for Fig. 1(b), $g_{11} > 0$ and $g_{12} > 0$, which together with $g_{22} = 34$, corresponding to repulsive atom-atom, atom-molecular, and molecule-molecule interactions.

As shown in this figure, different external potentials but with similar periodical structure can support the same stationary dark state solution. However, we want to point out the difference between Fig. 1(a) and Fig. 1(b). For Fig. 1(a), the external potential is relative shallow and the density distribution is mainly locating at the crest, which is the result of the competition between the nonlinear interaction and the external potential; while for Fig. 1(b), when the atom-atom and atom-molecule interaction become repulsive, the height of the external potential increases, and the density distributions of the order parameters are different from the former one and mainly locating at the wave trough.

Figure 2 shows the profiles of the atomic and molecular condensates $\phi_1(x)$ and $\phi_2(x)$ and external potential in the case of $g_{22} = 8$. In this case, $g_{11} = -0.5$ and $g_{12} = -4$, which are smaller than the parameters used in Fig. 1(a), but with a stronger molecule-molecule interactions. It is easy to see that the external potential presented here shows double-periodical structure, which can be realized by two pairs of counter-propagation lasers, and widely used in today's ultracold atom experiments. Finally, no matter the periodical or the double-periodical potentials, the density distribution always continuous distribution within the extension of the external potential. This is different from the localized solutions reported in [35] and the following solutions, where the

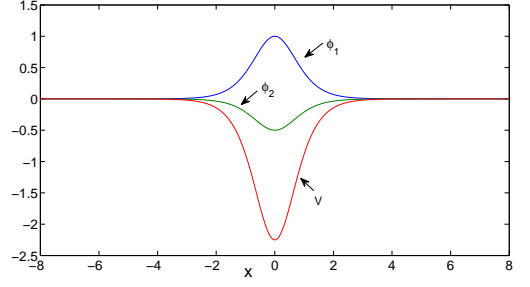


FIG. 4: (Color online) The stationary dark state wave functions of the atomic and molecular condensates $\phi_{13}(x)$ and $\phi_{23}(x)$ given by Eq. (16), and the profile of the external potential given by Eq. (17). The parameters are given as follows: $\delta = 1$, $\lambda = 1$, $a_0 = 0$, $\Omega = 1$, $A_0 = 1$, $g_{12} = 1$, $a_1 = 1$, $g_{22} = 20$, $\mu = -1$.

chemical potential must be smaller than zero.

Thus we conclude that when the ratio of the Rabi frequency is fixed, both periodical and double-periodical potentials can be applied to the coupled atomic-molecular Bose-Einstein condensates to get the analytical dark state solution via varying the strength of the nonlinear interactions.

Case 2. Now we shift our attention to the case of $k = 1$. In this situation, the stationary solutions for the atomic and molecular condensates can be written as

$$\begin{aligned}\phi_{12} &= \sqrt{\frac{a_0 B_1 \text{sech}^2(\lambda x)}{B_0 + B_1 \text{sech}^2(\lambda x)}}, \\ \phi_{22} &= \frac{-\Omega a_0 B_1 \text{sech}^2(\lambda x)}{2B_0 + 2B_1 \text{sech}^2(\lambda x)}.\end{aligned}\quad (12)$$

The relationships among the interaction parameters in this case read

$$\begin{aligned}g_{12} &= \frac{4\lambda^2(B_0 + B_1)}{\Omega^2 a_0^2 B_1} + g_{22}/2, \\ g_{11} &= \frac{2\lambda^2(B_0 + B_1)}{\Omega^2 a_0^2 B_1} - \frac{\lambda^2}{a_0} - \frac{B_0 \lambda^2}{2a_0 B_1} + \frac{g_{22}}{4},\end{aligned}\quad (13)$$

with g_{22} satisfying the following two cases

$$\begin{aligned}g_{22}^i &= \frac{4a_0 B_0 \lambda^2 \Omega^2 - 16\lambda^2 B_0 - 16B_1 \lambda^2 + 8a_0 \Omega^2 B_1}{B_1 a_0^2 \Omega^2 (2 + \Omega^2 a_0)}, \\ g_{22}^{ii} &= \frac{-8\lambda^2(B_0 + B_1)}{a_0^2 \Omega^2 B_1} - \frac{4B_1 \lambda^2 + 6B_0 \lambda^2}{a_0 B_1} + \frac{8}{a_0}.\end{aligned}\quad (14)$$

Shown in Fig. 3 are the wave functions of the atomic and molecular condensates $\phi_{12}(x)$ and $\phi_{22}(x)$, and the profile of external potential in the case of attractive molecule-molecule interaction. The parameters are given as follows: $\delta = 1$, $B_0 = B_1 = a_0 = 1$, $\lambda = 1$, $\Omega = -1$.

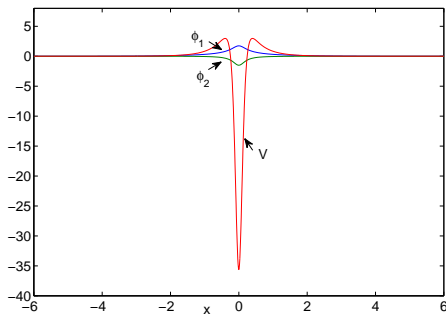


FIG. 5: (Color online) The same as in Fig. 4 but for $a_1 = 3$.

The molecule-molecule interaction $g_{22} = -20/3$ for Fig. 3(a) and $g_{22} = -18$ for Fig. 3(b), corresponding to the two different cases (i) and (ii) in Eq. (14), respectively. According to the constrain conditions for the nonlinear parameters, we can get $g_{11} = 5/6$, $g_{12} = 14/3$, and $g_{22} = -20/3$ for Fig. 3(a), which corresponding to repulsive atom-atom and atom-molecular interactions and attractive molecule-molecule interaction; while for Fig. 3(b), the nonlinear parameters read $g_{11} = -2$, $g_{12} = -1$, and $g_{22} = -18$, where all the interactions are attractive. As shown in Fig. 3(a), the external potential shows the harmonic structure, while double-well structure in Fig. 3(b), which is similar to the situation in Fig. (1): the order parameters $\phi_{12}(x)$ and $\phi_{22}(x)$ are with the same distributions in different potentials.

It is necessary to point out the stationary dark state solutions obtained in this case is different from the ones in *Case 1* as $\phi_{12}(x)$ and $\phi_{22}(x)$ are localized and symmetric with respect to certain point X (here we choose $X=0$). Thus we conclude that the exact localized dark state solutions can be achieved with a proper choice of the nonlinear interaction parameters and external potential, and both harmonic and double-well potentials can be used to trap the coupled atomic-molecular Bose-Einstein condensates to get the analytical dark state solution via varying the strength of the nonlinear interactions.

B. The Jacobi sn- and dn-wave solutions

In this subsection, we consider the Jacobi sn- and dn-function solution of $\varphi(x)$. In this case, one can easily get

$$\varphi_2 = A_0 + \frac{A_1 \text{sn}(\lambda x, k)^2}{\text{dn}(\lambda x, k)^2 + A_3}, \quad (15)$$

where sn and dn are the Jacobi elliptic functions and $0 \leq k \leq 1$ is the module of Jacobi elliptic functions. With the same procedure in subsection A, we can obtain a family of exact dark state solutions. As the expressions of wavefunctions for atomic and molecular condensates are too complicated, here we only focus on the limit $k = 1$ and list some main results.

In the case of $\delta = \lambda = \Omega = A_0 = g_{12} = 1$, $a_0 = 0, a_1 = 1$, $g_{22} = 20$, $\mu = -1$, the wave functions of the atomic and molecular condensates can be expressed as

$$\phi_{13} = \frac{\text{csgn}(1/\cosh(x))}{\cosh(x)}, \phi_{23} = \frac{-1}{2\cosh(x)^2}, \quad (16)$$

with csng the sign function for real and complex expressions. In this situation, the external potential is in the following form

$$V_2(x) = \frac{-1 - 8\cosh(x)^2}{4\cosh(x)^4}. \quad (17)$$

Figure 4 exhibits the profiles of the order parameters of the atomic and molecular condensates $\phi_{13}(x)$ and $\phi_{23}(x)$ and the external potential. As shown in this figure, the external potential is similar to the usual harmonic one and the result is similar to the one shown in Fig. (1c) in [35]; while Fig. 5 shows the same wave functions with the same parameters used in Fig. 4 but for $a_1 = 3$. It is easy to see that the external potential in Fig. 5 is different from the usual harmonic one, but with a peak in each side of the potential. It should be mentioned that in this case the nonlinear parameters are different. Thus we conclude that by manipulation the nonlinear parameters, there exist a class of external potentials where analytical dark solution can be obtained. Finally, we want to point out that the localized stationary dark state solutions and the corresponding two types of external potentials only exist for $\mu < 0$, which agrees well with the results obtained in [35].

The stability of the localized stationary solutions obtained in this paper is still an open problem. This can be achieved by carrying out numerical simulations of the localized stationary solutions with the imposed initial perturbations, or equivalently adopting the linear stability analysis: a small deviation away from the intended state is added to see the dynamical evolution of the system; if the deviation remains small, then the stability (or adiabaticity) is obeyed. Since the object of this work is to obtain some exact analytical solutions, a thorough analysis of the stability is beyond the scope of the present paper and these works belong to further studies.

IV. CONCLUSIONS

In summary, we focus on an atomic Bose-Einstein condensates coupled to a molecular condensates through the stimulated Raman adiabatic passage loaded in an external potential, with the emphasis on the stationary dark state solutions. Within the full parameter space accounts for atom-atom, atom-molecule and molecule-molecule collisions, together with the atom-molecule conversion coupling and the external potential, we detailed investigate under which conditions the exact dark state

solution can exist. Our results show that via suitable selections of the nonlinear parameter and external potential, the analytical dark state solutions can be constructed within a class of external potentials including double-well, periodical and double periodical, and harmonic potentials. The obtained results are of particular significance to experimental realization of such dark state in coupled atomic-molecular Bose-Einstein condensates.

V. ACKNOWLEDGMENTS

We would like to express our sincere thanks to Professor G. V. Shlyapnikov for helpful discussions.

This work was supported by NSFC under grants Nos. 10874235, 10934010, 60978019, the NKBRSCF under grants Nos. 2009CB930701, 2010CB922904, 2011CB921502, 2012CB821300, and NSFC-RGC under grants Nos. 11061160490 and 1386-N-HKU748/10.

-
- [1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, *Science* **269**, 198 (1995).
 - [2] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, *Phys. Rev. Lett.* **75**, 1687 (1995).
 - [3] K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).
 - [4] M. Greiner, C. A. Regal, and D. S. Jin, *Nature* **426**, 540 (2003).
 - [5] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, *Phys. Rev. Lett.* **92**, 120401 (2004).
 - [6] J. R. Abo-Shaeer, D. E. Miller, J.K. Chin, K. Xu, T. Mukaiyama, and W. Ketterle, *Phys. Rev. Lett.* **94**, 040405 (2005).
 - [7] M. Bartenstein, A. Altmeyer, S. Riedl, R. Geursen, S. Jochim, C. Chin, J. Hecker Denschlag, R. Grimm, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, *Phys. Rev. Lett.* **94**, 103201 (2005).
 - [8] J. J. Hudson, B. E. Sauer, M. R. Tarbutt, and E. A. Hinds, *Phys. Rev. Lett.* **89**, 023003 (2002).
 - [9] P. D. Drummond, K. V. Kheruntsyan, and H. He, *Phys. Rev. Lett.* **81**, 3055 (1998).
 - [10] E. Timmermans, P. Tommasini, R. Côté, M. Hussein, and A. Kerman, *Phys. Rev. Lett.* **83**, 2691 (1999).
 - [11] M. S. Chang, Q. S. Qin, W. X. Zhang, L. You, and M. S. Chapman, *Nature Phys.* **1**, 111 (2005).
 - [12] S. L. W. Midgley, S. Wüster, M. K. Olsen, M. J. Davis, and K. V. Kheruntsyan, *Phys. Rev. A* **79**, 053632 (2009).
 - [13] T. J. Alexander, E. A. Ostrovskaya, Y. S. Kivshar, and P. S. Julienne, *J. Opt. B* **4**, S33 (2002).
 - [14] M. Greiner, C. A. Regal, and D. S. Jin, *Nature (London)* **426**, 537 (2003).
 - [15] C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, *Nature (London)* **424**, 47 (2003).
 - [16] J. Cubizolles, T. Bourdel, S. J. J. M. F. Kokkelmans, G. V. Shlyapnikov, and C. Salomon, *Phys. Rev. Lett.* **91**, 240401 (2003).
 - [17] E. Timmermans, P. Tommasini, M. Hussein, and A. Kerman, *Phys. Rep.* **315**, 199 (1999).
 - [18] R. Wynar, R. S. Freeland, D. J. Han, and D. J. Heinzen, *Science* **287**, 1016 (2000).
 - [19] E. A. Donley, N. R. Claussen, S. T. Thomson, and C. E. Wieman, *Nature (London)* **417**, 529 (2002).
 - [20] B. Damski, L. Santos, E. Tiemann, M. Lewenstein, S. Kotochigova, P. Julienne, and P. Zoller, *Phys. Rev. Lett.* **90**, 110401 (2003).
 - [21] R. A. Duine and H. T. C. Stoof, *Phys. Rep.* **396**, 115 (2004).
 - [22] S. Dürr, T. Volz, A. Marte, and G. Rempe, *Phys. Rev. Lett.* **92**, 020406 (2004).
 - [23] H. Y. Ling, H. Pu, and B. Seaman, *Phys. Rev. Lett.* **93**, 250403 (2004).
 - [24] G. Thalhammer, K. Winkler, F. Lang, S. Schmid, R. Grimm, and J. H. Denschlag, *Phys. Rev. Lett.* **96**, 050402 (2006).
 - [25] T. Volz, N. Syassen, D. M. Bauer, E. Hansis, S. Dürr, and G. Rempe, *Nat. Phys.* **2**, 692 (2006).
 - [26] K. Winkler, G. Thalhammer, M. Theis, H. Ritsch, R. Grimm, and J. Hecker Denschlag, *Phys. Rev. Lett.* **95**, 063202 (2005).
 - [27] S. Inouye, M. R. Andrews, J. Stenger, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, *Nature (London)* **392**, 151 (1998).
 - [28] J. Stenger, S. Inouye, M. R. Andrews, H.-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, *Phys. Rev. Lett.* **82**, 2422 (1999).
 - [29] K. Winkler, F. Lang, G. Thalhammer, P. V. D. Straten, R. Grimm, and J. H. Denschlag, *Phys. Rev. Lett.* **98**, 043201 (2007).
 - [30] M. Mackie, R. Kowalski, and J. Javanainen, *Phys. Rev. Lett.* **84**, 3803 (2000).
 - [31] H. Pu, P. Maenner, W. Zhang, and H. Y. Ling, *Phys. Rev. Lett.* **98**, 050406 (2007).
 - [32] A. P. Itin and S. Watanabe, *Phys. Rev. Lett.* **99**, 223903 (2007).
 - [33] S. Y. Meng, L. B. Fu, and J. Liu, *Phys. Rev. A* **78**, 053410 (2008).
 - [34] X. F. Zhang, X. H. Hu, X. X. Liu, and W. M. Liu, *Phys. Rev. A* **79**, 033630 (2009).
 - [35] H. A. Cruz and V. V. Konotop, *Phys. Rev. A* **83**, 033603 (2011).
 - [36] G. Orso, L. P. Pitaevskii, S. Stringari, and M. Wouters, *Phys. Rev. Lett.* **95**, 060402 (2005).
 - [37] H. Y. Ling, P. Maenner, W. Zhang, and H. Pu, *Phys. Rev. A* **75**, 033615 (2007).